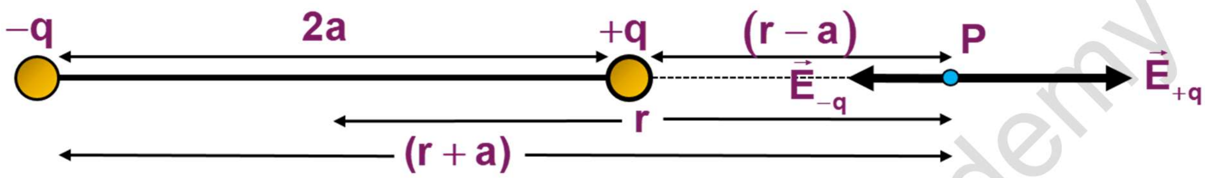
ELECTRIC CHARGES AND FIELDS – ALL DERIVATIONS

1. ELECTRIC FIELD AT A POINT ON AXIAL LINE

Consider an electric dipole consisting of two charges +q and – q as shown. We have to find electric field due to this dipole at a point P on axial line at distance r from the centre of this dipole. Clearly, the distance of P from – q is (r + a) and from +q is (r – a).



Electric field at P due to +q is

$$E_{+q} = \frac{q}{4\pi\varepsilon_o}\frac{1}{(r-a)^2}$$

And electric field at P due to – q is

$$E_{-q} = \frac{q}{4\pi\varepsilon_o}\frac{1}{(r+a)^2}$$

Therefore, net field at P is

$$E_{axial} = E_{+q} - E_{-q} = \frac{q}{4\pi\varepsilon_o}\left[\frac{1}{(r-a)^2} - \frac{1}{(r+a)^2}\right]$$

$$\Rightarrow \frac{q}{4\pi\varepsilon_o}\left[\frac{(r+a)^2-(r-a)^2}{(r^2-a^2)^2}\right]$$

$$\Rightarrow \frac{q}{4\pi\varepsilon_o}\left[\frac{(r^2+a^2+2ar)-(r^2+a^2-2ar)}{(r^2-a^2)^2}\right]$$

$$\Rightarrow \frac{2aq(2r)}{4\pi\varepsilon_o(r^2-a^2)^2}$$

$$\Rightarrow \boxed{\frac{2pr}{4\pi\varepsilon_o(a^2-r^2)^2}} \qquad [p = 2aq]$$

$$\Rightarrow \frac{q}{4\pi\varepsilon_o}\left[\frac{\left(r^2+a^2+2ar-r^2-a^2+2ar\right)}{\left(r^2-a^2\right)^2}\right]$$

$$\Rightarrow \frac{q}{4\pi\varepsilon_o}\frac{4ar}{\left(r^2-a^2\right)^2}$$

For short dipole $r >> a$

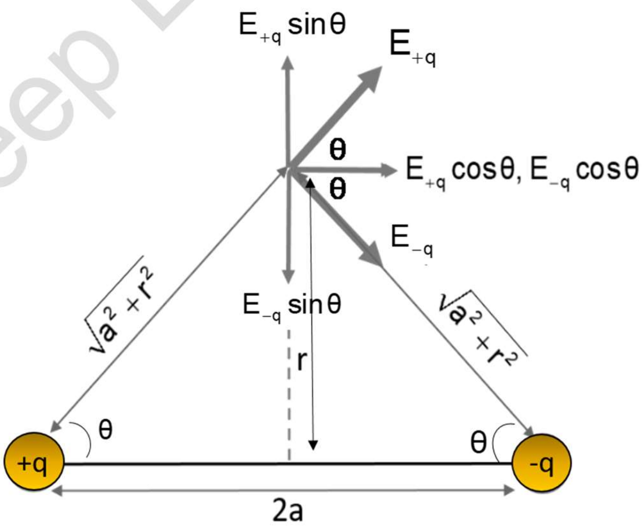
$$E=\frac{2rp}{4\pi\varepsilon_o r^4}$$

$$\Rightarrow \boxed{E=\frac{2p}{4\pi\varepsilon_o r^3}}$$

2. ELECTRIC FIELD AT A POINT ON EQUATORIAL LINE

Consider an electric dipole consisting of two charges +q and – q as shown. We have to find electric field due to this dipole at a point P on axial line at distance r from the centre of this dipole. Clearly, the distance of P from – q is (r + a) and from +q is (r – a).

Due to symmetry electric field at P due to both +q and – q will be same which is given by



$$E_{-q}=E_{+q}=\frac{q}{4\pi\varepsilon_o\left(r^2+a^2\right)}$$

The directions of E_{-q} and E_{+q} are as shown in the figure. The components normal to the dipole axis $\left(E_{+q}\sin\theta \text{ and } E_{-q}\sin\theta\right)$ cancel out and $\left(E_{+q}\cos\theta \text{ and } E_{-q}\cos\theta\right)$ will add up

$$E_{eq} = E_{+q}\cos\theta + E_{-q}\cos\theta$$

$$\Rightarrow E_{eq} = 2E_{+q}\cos\theta$$

$$\Rightarrow E_{eq} = \frac{2q}{4\pi\varepsilon_o\left(r^2+a^2\right)}\cos\theta$$

$$\Rightarrow E_{eq} = \frac{2q}{4\pi\varepsilon_o\left(r^2+a^2\right)} \times \frac{a}{\sqrt{r^2+a^2}}$$

$$\Rightarrow E_{eq} = \frac{2aq}{4\pi\varepsilon_o\left(r^2+a^2\right)^{\frac{3}{2}}}$$

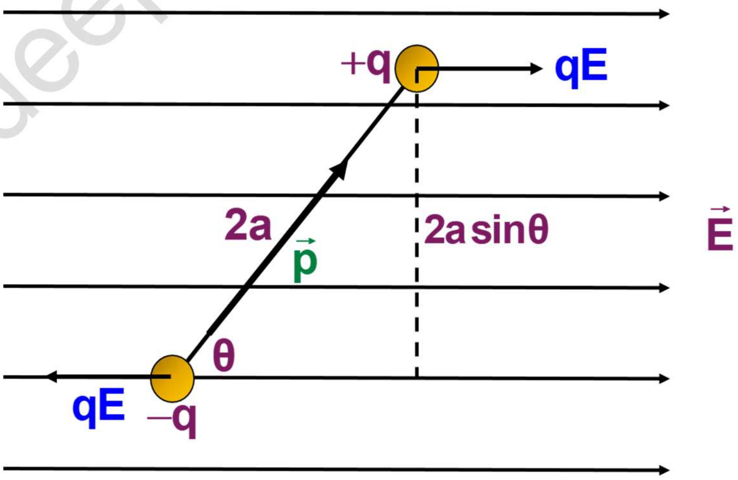
$$\Rightarrow \boxed{\frac{p}{4\pi\varepsilon_o\left(r^2+a^2\right)^{\frac{3}{2}}}}$$

For short dipole $r >> a$

$$\Rightarrow \boxed{E = \frac{p}{4\pi\varepsilon_o r^3}}$$

3. TORQUE ACTING ON AN ELECTRIC DIPOLE IN EXTERNAL ELECTRIC FIELD

Consider an electric dipole consisting pf charges – q and + q and of length 2a placed in a uniform electric field making an angle θ with electric field.



Force on charge – q = $-qE$ acting opposite to the field

Force on charge +q = qE acting along the field

Electric dipole is under the action of two equal and unlike parallel forces, which give rise to a torque on the dipole.

$$\tau = \text{Force} \times \text{perpendicular distance between the two forces}$$
$$\tau = qE \times AN$$
$$\Rightarrow \tau = qE \times 2a\sin\theta$$
$$\Rightarrow \tau = (q \times 2a)E\sin\theta$$
$$\Rightarrow \tau = pE\sin\theta$$
$$\Rightarrow \boxed{\vec{\tau} = \vec{p} \times \vec{E}}$$

SPECIAL CASES

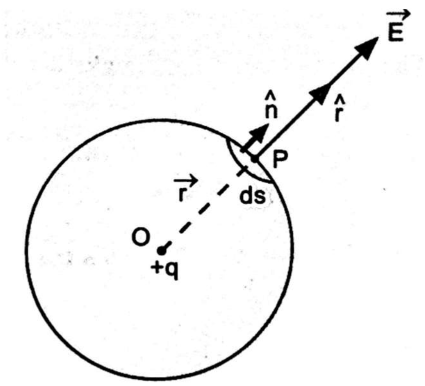
1. If $\theta = 0^\circ, \sin 0^\circ = 0, \therefore \tau = 0$, this condition is called stable equilibrium. When the dipole is displaced from this orientation it always comes back to the same configuration.
2. $\theta = 180^\circ, \sin 180^\circ = 0, \therefore \tau = 0$

   this condition is called unstable equilibrium because once displaced the dipole never comes back to this orientation instead it aligns itself parallel to the field.
3. $\theta = 90^\circ, \sin 90^\circ = 1, \therefore \tau = pE$ (maximum)

Please note: - In a non-uniform electric field $F_{net} \neq 0, \tau \neq 0$, therefore dipole executes both translation and oscillation.

4. DERIVATION OF COULOMB'S LAW FROM GAUSS LAW

Consider an isolated positive point charge q at O. Imagine a sphere of radius r with centre O. the magnitude of electric field intensity **E** at every point on the surface is the same and it is directed radially outwards.



Therefore, the direction of vector **ds** representing a small area element on the surface of sphere is along E only i.e. $\theta = 0^\circ$.

According to Gauss law

$$\oint \vec{E}.\vec{ds} = \frac{q}{\varepsilon_o}$$

$$\Rightarrow \oint Eds\cos 0^\circ = \frac{q}{\varepsilon_o}$$

$$\Rightarrow E\oint ds\cos 0^\circ = \frac{q}{\varepsilon_o}$$

$$\Rightarrow E\left(4\pi r^2\right) = \frac{q}{\varepsilon_o}$$

$$\Rightarrow E = \frac{q}{4\pi r^2}$$

This is the electric field intensity at any point P distant r from an isolated point charge q at the centre of the sphere. If another point charge q_o were placed at point P, then force on q_o would be

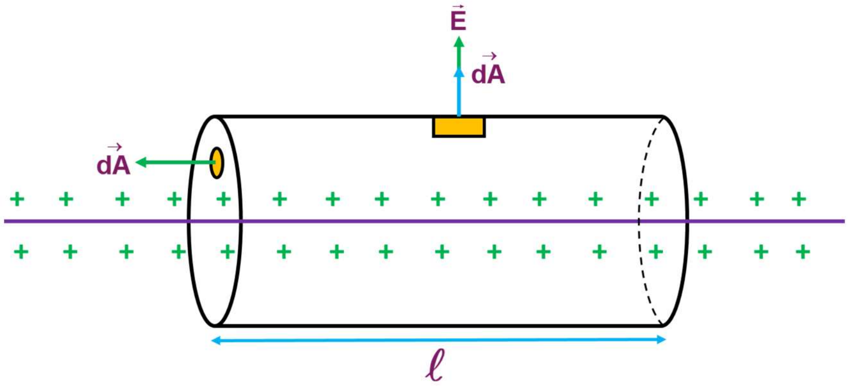
$$F = q_o \times E$$

$$\Rightarrow F = \frac{qq_o}{4\pi\varepsilon_o r^2}$$

Which is Coulomb's law.

5. ELECTRIC FIELD DUE TO A STRAIGHT LONG CHARGED CONDUCTOR

Consider a straight charged conductor of length *l* as shown. Consider a cylindrical Gaussian surface of radius r around this conductor. Let ds be small area on this surface. As conductor is positively charged, electric field due it is outwards. Therefore, electric field and area vector are in same direction. Applying gauss theorem, we have



$$\oint \vec{E}.\vec{ds} = \frac{q}{\varepsilon_o}$$

$$\Rightarrow \oint Eds\cos 0^\circ = \frac{q}{\varepsilon_o}$$

$$\Rightarrow \oint Eds = \frac{q}{\varepsilon_o}$$

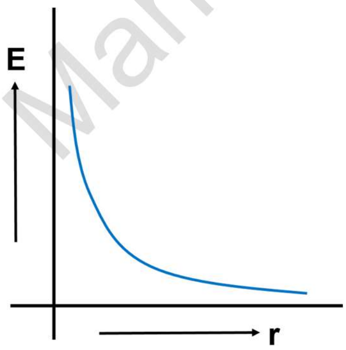
$$\Rightarrow E(2\pi rl) = \frac{q}{\varepsilon_o}$$

$$\Rightarrow E = \frac{q}{2\pi\varepsilon_o rl}$$

$$\Rightarrow \boxed{E = \frac{\lambda}{2\pi\varepsilon_o r}}$$

$\lambda = \frac{q}{l}$, λ is called linear charge density

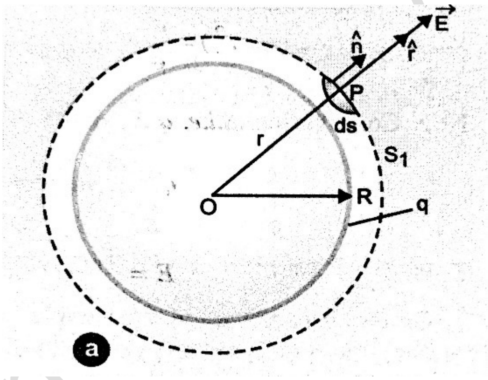
Clearly, $E \propto \frac{1}{r}$. Therefore, the variation of E with r is shown graphically in the figure shown below:



6. ELECTRIC FIELD DUE TO SPHERICAL SHELL

When point P lies outside the spherical shell

Suppose that we have to calculate electric field at the point P at a distance *r* (*r* > *R*) from its centre. Draw the Gaussian surface through point P so as to enclose the charged spherical shell. The Gaussian surface is a spherical shell of radius *r* and centre O.



Let $\vec{E}$ be the electric field at point P. Then, the electric flux through area element $\vec{ds}$ is given by,

$$d\varphi_E = \vec{E}.\vec{ds}$$

Since $\vec{ds}$ is also along normal to the surface,

$$d\varphi = Eds\cos 0^\circ = Eds$$

∴ Total electric flux through the Gaussian surface is given by,

$$\varphi = \oint Eds = E\oint ds$$

Now,

$$\oint ds = 4\pi r^2$$

$$\therefore \varphi = E \times 4\pi r^2 \quad ....(i)$$

Since the charge enclosed by the Gaussian surface is *q*, according to Gauss theorem,

$$\varphi = \frac{q}{\varepsilon_o} \quad ....(ii)$$

From equations (i) and (ii), we obtain

$$E \times 4\pi r^2 = \frac{q}{\varepsilon_o}$$

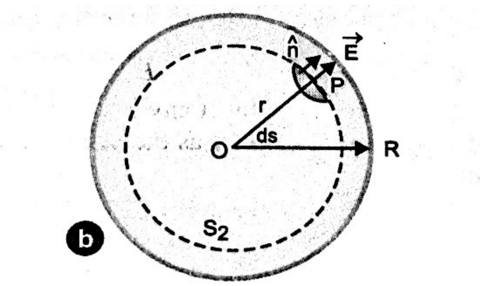
$$\Rightarrow E = \frac{q}{4\pi\varepsilon_o r^2} \qquad (\text{for } r > R)$$

When point P lies inside the spherical shell

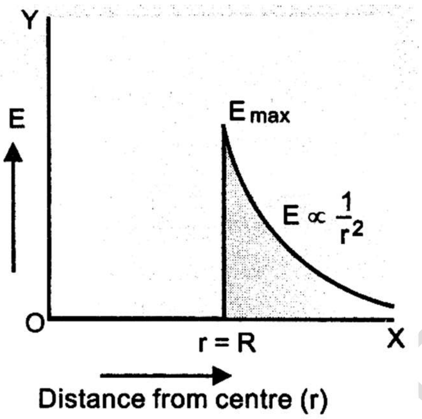
In such a case, the Gaussian surface encloses no charge.

According to Gauss law,

$$E \times 4\pi r^2 = 0$$
$$\Rightarrow E = 0$$



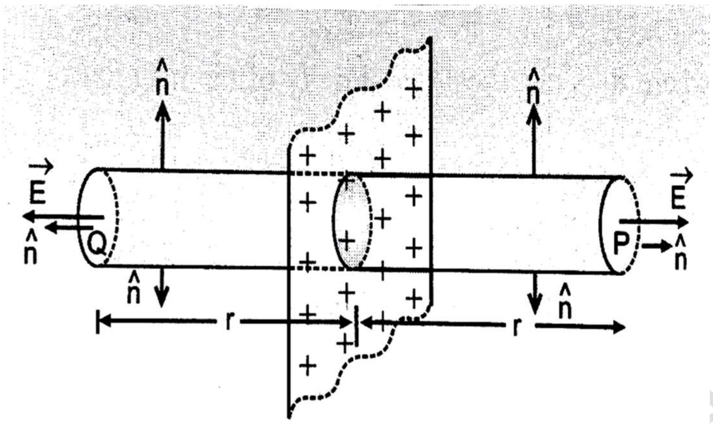
Hence, the field due to a uniformly charged spherical shell is zero at all points inside the shell. The variation of electric field intensity E with distance from the centre of a uniformly charge spherical shell is shown:



7. ELECTRIC FIELD DUE TO INFINITE PLANE SHEET OF CHARGE

Consider an infinite thin plane sheet of positive charge having a uniform surface charge density σ on both sides of the sheet.

Let P be the point at a distance ‘*a*’ from the sheet at which electric field is required.



Draw a Gaussian cylinder of area of cross-section A through point P. The electric flux crossing through the Gaussian surface is given by,

Φ = E × Area of the circular caps of the cylinder

Since electric lines of force are parallel to the curved surface of the cylinder, the flux due to electric field of the plane sheet of charge passes only through the two circular caps of the cylinder.

$\varphi = E \times 2A$ ......(i)

According to Gauss theorem, we have

$$\varphi = \frac{q}{\varepsilon_o}$$

Here, the charge enclosed by the Gaussian surface,

$q = \sigma A$ where σ is the surface charge density (q/A)

$$\therefore \varphi = \frac{\sigma A}{\varepsilon_o} \quad ....(ii)$$

From equations (i) and (ii), we obtain

$$E \times 2A = \frac{\sigma A}{\varepsilon_o} \Rightarrow \boxed{E = \frac{\sigma}{2\varepsilon_o}}$$

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